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# Adapting coordinates

Joost Berson*, Jan Willem Bikker, Arno van den Essen

*Department of Mathematics, University of Nijmegen, Toernooiveld, Nijmegen 6525 ED, The Netherlands*

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**Abstract**

In this paper, we extend and generalize some work by Edo and Vénéreau concerning the question how to recognize coordinates over a commutative ring. More precisely, it is shown that it is often sufficient to look at quotients and localizations of the given ring, for which the question becomes more tractable.


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## 1. Introduction

Let $A$ be a commutative ring. A polynomial $f \in A[x] := A[x_1,\dots,x_n]$ is called a coordinate in $A[x]$ if there exist $f_2,\dots,f_n \in A[x]$ such that $A[f,f_2,\dots,f_n] = A[x]$. Recognizing coordinates is one of the most important (and difficult) problems in the study of polynomial automorphisms. In case $n=2$ and $A$ a field this problem was solved by various authors. See for example [4,5]. Studying coordinates over rings which are not fields is much more complicated. For example the polynomial $y + z^2x + zy^2$ is a coordinate in $A[x,y]$, where $A := \mathbb{C}[z]$; in fact this polynomial is the second component of the well-known Nagata automorphism [12]. Such coordinates were first investigated by Nagata [12] and later by Drensky and Yu [7] and Edo and Vénéreau [8]. One of the results of this last paper is the following: if $a$ is a non-zerodivisor in $A$ and $g(y) \in A[y]$, then $za + g(y)$ is a coordinate in $A[y,z]$ iff $\bar{g}(y)$ is a coordinate in $\bar{A}[y]$, where $\bar{A} := A/Aa$ and $\bar{g}$ is obtained from $g$ by reducing its coefficients modulo the ideal $Aa$. This result was generalized by Vénéreau [14]. His result asserts the following: if $f(y,z)$ is a coordinate in $A[y,z]$ and $a$ a non-zerodivisor in $A$ such that $\bar{f}(y,0)$ is a

* Corresponding author.
*E-mail address:* berson@sci.kun.nl (J. Berson).


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coordinate in $\bar{A}[y]$ then $f(y,az)$ is a coordinate in $A[y,z]$ over $A$. Indeed the result of Edo and Vénéreau [8] mentioned earlier follows from this result by taking $f=z+g(y)$, which is obviously a coordinate in $A[y,z]$.

In this paper we study various extensions and generalizations of this result. First of all, we show that the condition '$a$ is non-zerodivisor in $A$' is superfluous (3.1). In fact, the proof given can be readily extended to coordinate systems of co-length one (3.2 and 3.3).

In Section 4, we improve upon Vénéreaus theorem, in case $A$ is a $\mathbb{Q}$-algebra, by showing the following: let $a \in A$ be arbitrary and $f \in A[y,z]$. If $f$ is a coordinate when considered in $(A/Aa)[y,z]$ and also when considered in $A_a[y,z]$, then $f$ is a coordinate in $A[y,z]$. The proof uses a special result concerning polynomials in two variables, namely the fact that $f$ is a coordinate iff it is a so-called residual coordinate, i.e. a coordinate considered in $k_\wp[y,z]$ for every prime ideal $\wp$ of $A$, where $k_\wp$ is the residue field $A_\wp/\wp A_\wp$. In order to generalize 4.1 to partial coordinate systems of co-length one we give another proof of this result. A crucial ingredient in the proof is the so-called determinant-one theorem (4.3). We would like to attract the attention of the reader to this result in connection with the study of the Jacobian Conjecture (see [9, Section 2.3]). Finally, in Section 5 we study conditions on the ring $A$ which guarantee that if $p(y_1,y_2,z)$ is a coordinate in $A[y_1,y_2,z]$ and $\bar{p}(y_1,y_2,0)$ is a coordinate in $\bar{A}[y_1,y_2]$, where $\bar{A} := A/Aa$, then also $p(y_1,y_2,az)$ is a coordinate in $A[y_1,y_2,z]$, thereby generalizing some other results of Vénéreau [15].

## 2. Preliminaries

In this section, we collect some more or less well-known results concerning coordinates and locally nilpotent derivations. Throughout this paper, $A$ is a commutative ring and $A[x] := A[x_1,\dots,x_n]$ is the polynomial ring in $n$ variables over $A$. Sometimes we denote this ring by $A^{[n]}$.

**Definition 2.1.** A set of $n$ polynomials $F_1,\dots,F_n$ in $A[x]$ is called a *coordinate system* of $A[x]$ if $A[F_1,\dots,F_n] = A[x]$. If $1 \leqslant k < n$ a sequence $F_1,\dots,F_k$ of polynomials is called a *partial coordinate system* if there exist $F_{k+1},\dots,F_n$ such that $A[F_1,\dots,F_k, F_{k+1},\dots,F_n]=A[x]$. If $k=1$ in this situation, $F_1$ is called a *coordinate* in $A[x]$ (over $A$) and if $k=n-1$ we say that $F_1,\dots,F_{n-1}$ is a partial coordinate system of *co-length one*.

**Remark 2.2.** Let $I$ be an ideal contained in the nilradical of $A$. Then it is well known (see for example [9, Lemma 1.1.9]) that a polynomial map $F=(F_1,\dots,F_n) \in A[x]^n$ is invertible iff $\bar{F}$ is invertible over $\bar{A} := A/I$. From this it follows that a sequence $(F_1,\dots,F_k) \in A[x]^k$ (where $1 \leqslant k \leqslant n$) is a partial coordinate system iff it is a partial coordinate system over $\bar{A}$. We will use these facts in this paper without further reference.

**Example 2.3.** Let $n=1$ and write $x$ instead of $x_1$. Let $f = a_0 + a_1x + \cdots + a_nx^n$ for some $a_i$ in $A$. Then $f$ is a coordinate in $A[x]$ iff $a_1$ is a unit in $A$ and $a_i$ is nilpotent for each $i \geqslant 2$. This can be seen as follows. Suppose that $f$ is a coordinate with inverse $g \in A[x]$, say $\deg(f)=n$, $\deg(g)=m$. If $A$ is a domain, then $\deg(x)=\deg(g(f(x)))=nm$, therefore $\deg(f)=1$. Now let $A$ be any ring. Since $f$ is linear modulo every prime ideal, the $a_i$ with $i \geqslant 2$ belong to every prime ideal, meaning that they are nilpotent. Furthermore, $\det J(f)(0) = a_1 \in A^*$. The converse is easy.

Now assume that $A$ is a $\mathbb{Q}$-algebra and consider the polynomial ring in two variables $x$ and $y$ over $A$, i.e. $A[x, y]$. For every prime ideal $\wp$ of $A$ put $k_\wp := A_\wp/\wp A_\wp$ (which is isomorphic to the quotient field of $A/\wp$).

For $f \in A[x, y]$ we denote the corresponding element of $k_\wp[x, y]$ also by $f$. The following result was proved in [3,10].

**Theorem 2.4.** *Let $A$ be a $\mathbb{Q}$-algebra and $f \in A[x, y]$. Then $f$ is a coordinate in $A[x, y]$ iff $f$ is a coordinate in $k_\wp[x, y]$ for all prime ideals $\wp$ of $A$.*

Another important result which is an immediate consequence of the Abhyankar–Moh theorem (which holds for fields containing $\mathbb{Q}$) and Theorem 2.4 is

**Corollary 2.5** (Generalized Abhyankar–Moh theorem). *Let $A$ be a $\mathbb{Q}$-algebra and $f \in A[x, y]$. Then the following statements are equivalent*:
(1) $A[x, y]/(f) \cong_A A^{[1]}$ (*isomorphism of $A$-algebras*),
(2) $f$ *is a coordinate of* $A[x, y]$ (*over* $A$).

We will also need the following result concerning (in particular) coordinates in $n$ variables (for a proof see [6]). Recall that a ring $A$ is called a *Hermite ring* if every unimodular row in $A$ (i.e. a row $(a_1, \ldots, a_m)$ with all $a_i \in A$ such that $Aa_1 + \cdots + Aa_m = A$) can be extended to an invertible matrix in $\mathrm{GL}_m(A)$.

**Theorem 2.6.** *Let $A$ be a Hermite ring and a domain, $1 \leqslant k \leqslant n-1$ and $h_1, \ldots, h_k$ a partial coordinate system of $A_\mathfrak{m}[x]$ for every maximal ideal $\mathfrak{m}$ of $A$, then $(h_1, \ldots, h_k)$ is a partial coordinate system of $A[x]$.*

## 3. An extension of a result of Vénéreau

The aim of this section is to prove the following result and its corollaries.

**Proposition 3.1.** *Let $A$ be any ring and $a \in A$ arbitrary. Let $(p(y,z), f(y,z)) \in \mathrm{Aut}_A A[y,z]$. If $\bar{p}(y,0) \in \mathrm{Aut}_{\bar{A}} \bar{A}[y]$, where $\bar{A} := A/Aa$, then $p(y, az)$ is a coordinate in $A[y,z]$.*

*More precisely, if $\bar{p}(y,0) \in \mathrm{Aut}_{\bar{A}} \bar{A}[y]$ then there exist $q(y)$ in $A[y]$ and $u(y,z)$, $v(y,z)$ in $A[y,z]$ such that $f(y,z) = q(p(y,z)) + a \cdot u(y,z) + z \cdot v(y,z)$. Furthermore, $F = (p(y,az), f'(y,z)) \in \mathrm{Aut}_A A[y,z]$, where $f'(y,z) := u(y,az) + zv(y,az)$.*

**Proof.** (i) By the hypothesis on $\bar{p}(y,0)$ there exists $q(y) \in A[y]$ such that $f(y,0) \equiv q(p(y,0)) \bmod a$, which implies the existence of $u$ and $v$. Now take $f'$ and $F$ as in the proposition. We need to show that $F$ is invertible over $A$. It is well known that it suffices to show that for each prime ideal $\wp$ of $A$, $\tilde{F}$ is invertible over $\tilde{A}$, where $\tilde{A} := A/\wp$ (see for example [9, pp. 6, 7] or [13, 3.5.4]). Now we distinguish two cases: $a \in \wp$ and $a \notin \wp$.

(ii) Case $a \in \wp$. Then $\tilde{a} = 0$ in $\tilde{A}$ and we have to prove that $\tilde{F} = (\tilde{p}(y,0), \tilde{f}'(y,z))$ is invertible over $\tilde{A}$. Since $\tilde{A}[y] = \tilde{A}[\tilde{p}(y,0)]$ there exists $r(y) \in A[y]$ with $y = \tilde{r}(\tilde{p}(y,0))$. Therefore, $\tilde{F}$ is invertible over $\tilde{A}$ iff $(\tilde{r}(y),z) \circ \tilde{F} = (y, \tilde{f}'(y,z)) = (y, \tilde{u}(y,0) + z\tilde{v}(y,0))$ is invertible over $\tilde{A}$ iff $\tilde{v}(y,0) \in \tilde{A}^*$. Indeed, $\tilde{v}(y,0) \in \tilde{A}^*$ since $(\tilde{p}(y,z), \tilde{f}(y,z)) = (\tilde{p}(y,z), \tilde{q}(\tilde{p}(y,z)) + z\tilde{v}(y,z))$ is invertible over $\tilde{A}$, which implies that $(\tilde{p}(y,z), z\tilde{v}(y,z))$ is invertible over $\tilde{A}$ and hence that $z\tilde{v}(y,z)$ is irreducible over the domain $\tilde{A}$. (Here we use the fact that coordinates over a domain are irreducible, which is not true for general rings.) Consequently $\tilde{v}(y,0)$ is a unit in $\tilde{A}$.

(iii) Case $a \notin \wp$. Then $\tilde{A}$ is a domain and $\tilde{a} \neq 0$. So it suffices to prove our proposition for the case that $A$ is a domain and $a \neq 0$. Put $d := \det J(p,f)$. Since $(p,f) \in \mathrm{Aut}_A A[y,z]$ and $A$ is a domain, it follows that $d \in A^*$ and that $F^{(1)} := (p(y,az), f(y,az))$ is invertible over $Q(A)$ with $\det JF^{(1)} = ad$. So $F^{(2)} := (p(y,az), f(y,az) - q(p(y,az)))$ is invertible over $Q(A)$, also with $\det JF^{(2)} = ad$. Now observe that $F = (y, z/a) \circ F^{(2)}$. So $F$ is invertible over $Q(A)$ and $\det JF = (1/a)ad = d \in A^*$. Then it follows from Lemma 1.1.8 in [9] that $F$ is invertible over $A$. $\square$

Now we show how this result can be extended to partial coordinate systems of co-length one.

**Corollary 3.2.** *Let $y = (y_1, \ldots, y_n)$ and $(p_1(y,z), \ldots, p_n(y,z), f) \in \mathrm{Aut}_A A[y,z]$. If $(\bar{p}_1(y,0), \ldots, \bar{p}_n(y,0)) \in \mathrm{Aut}_{\bar{A}} \bar{A}[y]$, where $\bar{A} = A/Aa$, then $(p_1(y,az), \ldots, p_n(y,az))$ is a partial coordinate system of co-length one of $A[y,z]$.*

**Proof.** Just replace everywhere in the proof of 3.1 '$y$' by $(y_1, \ldots, y_n)$ and '$p$' by $(p_1, \ldots, p_n)$. $\square$

**Corollary 3.3.** *Let $y = (y_1, \ldots, y_n)$ and $(p_1(y,z), \ldots, p_n(y,z), f) \in \mathrm{Aut}_A A[y,z]$. Let $\bar{A} = A/Aa$. Then $(p_1(y,az), \ldots, p_n(y,az))$ is a partial coordinate system of co-length one of $A[y,z]$ iff $(\bar{p}_1(y,0), \ldots, \bar{p}_n(y,0)) \in \mathrm{Aut}_{\bar{A}} \bar{A}[y]$.*

**Proof.** The 'if'-part is 3.2. Conversely, suppose that $(p_1(y,az), \ldots, p_n(y,az))$ is a partial coordinate system of $A[y,z]$ of co-length one, say $(p(y,az), h(y,z)) \in \mathrm{Aut}_A A[y,z]$. Then modulo $a$ it is also an automorphism of $\bar{A}[y,z]$ over $\bar{A}$. Let $(g_1(y,z), \ldots, g_n(y,z), s(y,z))$ be the inverse of $(\bar{p}(y,0), \bar{h}(y,z))$ over $\bar{A}$. Then $y_i = \bar{p}_i(g(y,z),0) = \bar{p}_i(g(y,0),0)$ for all $i$. So $(g_1(y,0), \ldots, g_n(y,0)) \in \mathrm{Aut}_{\bar{A}} \bar{A}[y]$ with inverse $(\bar{p}_1(y,0), \ldots, \bar{p}_n(y,0)) \in \mathrm{Aut}_{\bar{A}} \bar{A}[y]$. $\square$

## 4. Coordinates modulo and localized in an element

Throughout this section we assume that $A$ is a $\mathbb{Q}$-algebra. We start by generalizing 3.1 in the case of two variables.

**Proposition 4.1.** *Let $f \in A[y,z]$ and $a \in A$. If $f$ is a coordinate when considered in $A_a[y,z]$ and also when considered in $(A/Aa)[y,z]$, then $f$ is a coordinate in $A[y,z]$.*

**Proof.** Let $\wp$ be a prime ideal of $A$. If $a \in \wp$ it follows that $f$ is a coordinate in $A/\wp$ and hence in $k_\wp[y,z]$. If $a \notin \wp$ the natural homomorphism $A_a \to A_\wp$ implies that $f$ is a coordinate in $A_\wp[y,z]$ and hence in $k_\wp[y,z]$. Then apply Theorem 2.4. □

In order to generalize 4.1 to the case of a partial coordinate system of co-length, one we will give another proof. However, the price we have to pay is that we have to assume that $a$ is a non-zerodivisor in $A$. More precisely we have

**Theorem 4.2.** *Let $a \in A$ be a non-zerodivisor. Let $y = (y_1, \ldots, y_n)$. If $p(y,z) := (p_1(y,z), \ldots, p_n(y,z))$ is a partial coordinate system when considered over the ring $A_a$ and also when considered over the ring $A/Aa$, then $p(y,z)$ is a partial coordinate system over $A$.*

The proof of this result makes use of the following result which, as observed in the introduction, is interesting in itself since it may be useful in the study of the Jacobian Conjecture (see [9, Section 2.3]).

**Theorem 4.3** (Determinant-one theorem). *Let $F \in A[x]^n$ be an invertible polynomial map, where $A[x] := A[x_1, \ldots, x_n]$. Then there exists $s \in A[x]$ such that $(F_1, \ldots, F_{n-1}, s)$ is invertible over $A$ and $\det J(F_1, \ldots, F_{n-1}, s) = 1$.*

**Proof.** Let $d := \det JF \in A[x]^*$. So $d(0) \in A^*$ and if $\eta$ denotes the ideal generated by the (finite set of) coefficients of $d$, except $d(0)$, then it is an ideal contained in the nilradical of $A$ and there exists $N \in \mathbb{N}$ with $\eta^N = 0$. We will write $\bar{A} := A/\eta$.

Let $D$ be the $A$-derivation on $A[x]$ defined by $D(h) := \det J(F_1, \ldots, F_{n-1}, h)$ for all $h \in A[x]$. Let $P_0^{(0)} := F_n \cdot d(0)^{-1}$, then $(\bar{F}_1, \ldots, \bar{F}_{n-1}, \bar{P}_0^{(0)})$ is invertible over $\bar{A}$ and $D(P_0^{(0)}) \equiv 1 \bmod \eta$. Because $\bar{A}[x_1, \ldots, x_n] = \bar{A}[\bar{F}_1, \ldots, \bar{F}_{n-1}, \bar{P}_0^{(0)}] = \bar{A}^{[n]}$ we can view $\bar{D}$ as the partial derivative with respect to the polynomial variable $\bar{P}_0^{(0)}$. Therefore, $\bar{D}$ is surjective and for any multidegree $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we can find a $P_\alpha^{(0)} \in A[x]$ such that $D(P_\alpha^{(0)}) \equiv x^\alpha \bmod \eta$ (where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$). With induction, we will make for all $\alpha \in \mathbb{N}^n$ and for all $k \in \mathbb{N}^*$ a $P_\alpha^{(k)}$ such that $D(P_\alpha^{(k)}) \equiv x^\alpha \bmod \eta^{2^k}$ and also $P_\alpha^{(k)} \equiv P_\alpha^{(0)} \bmod \eta$. Namely, if we have this then we can choose $k$ such that $2^k \geqslant N$, which implies that $\eta^{2^k} = (0)$ so $D(P_0^{(k)}) = 1$. Moreover $(F_1, \ldots, F_{n-1}, P_0^{(k)})$ is invertible, because modulo $\eta$ it is equal to $(F_1, \ldots, F_{n-1}, P_0^{(0)})$, which is invertible because it has this property modulo $\eta$, a part of the nilradical. Therefore $s = P_0^{(k)}$ works.

It remains to show the induction. This is a generalization of Lemma 2.5 of [2]. Suppose for $k \in \mathbb{N}^*$ we have for all $\alpha \in \mathbb{N}^n$ a $P_\alpha^{(k)}$ with $D(P_\alpha^{(k)}) \equiv x^\alpha \bmod \eta^{2^k}$ and $P_\alpha^{(k)} \equiv P_\alpha^{(0)} \bmod \eta$. Write $D(P_\alpha^{(k)}) = x^\alpha + \sum_\beta c_{\alpha\beta} x^\beta$, with $c_{\alpha\beta} \in \eta^{2^k}$. We define

$$P_\alpha^{(k+1)} := P_\alpha^{(k)} - \sum_\gamma c_{\alpha\gamma} P_\gamma^{(k)},$$

then

$$
\begin{aligned}
D(P_\alpha^{(k+1)}) &= D(P_\alpha^{(k)}) - \sum_\gamma c_{\alpha\gamma} D(P_\gamma^{(k)}) \\
&= x^\alpha + \sum_\beta c_{\alpha\beta} x^\beta - \sum_\gamma c_{\alpha\gamma} \cdot \left( x^\gamma + \sum_\delta c_{\gamma\delta} x^\delta \right) \\
&= x^\alpha - \sum_\gamma \sum_\delta c_{\alpha\gamma} c_{\gamma\delta} x^\delta \equiv x^\alpha \bmod \eta^{2^{k+1}}
\end{aligned}
$$

since $c_{\alpha\gamma}$ and $c_{\gamma\delta}$ are in $\eta^{2^k}$.

Furthermore, by construction $P_\alpha^{(k+1)} \equiv P_\alpha^{(k)} \equiv P_\alpha^{(0)} \bmod \eta$. □

Now we can give the proof of Theorem 4.2.

**Proof.** Since $p(y,z)$ is a partial coordinate system of $A_a[y,z]$ there exists $\tilde{H} \in A_a[y,z]$ such that $(p,\tilde{H}) \in \mathrm{Aut}_{A_a} A_a[y,z]$. Using the determinant-one Theorem 4.3 we may assume that $\det J(p,\tilde{H}) = 1$. Now take $m \in \mathbb{N}$ and choose $H \in A[y,z]$ such that $H = a^m \tilde{H}$ (in $A_a[y,z]$). Then $\det J(p,H) = a^m \in A_a^*$ and $(p,H)$ is invertible over $A_a$. Furthermore, $p$ is a partial coordinate system modulo $(a)$, but even modulo $(a^m)$. Namely, $b := a+(a^m)$ is an element of the nilradical of $A/(a^m)$, and $(A/(a^m))/(b) \cong A/(a)$. Using the determinant-one Theorem 4.3 there exists $G \in A[y,z]$ such that $(\bar{p},\bar{G})$ is invertible over $A/(a^m)$ and $\det J(p,G) \equiv 1 \bmod a^m$. Write $\bar{A} := A/(a^m)$ and define the $A$-derivation $D$ on $A[y,z]$ by $D(h) := \det J(p_1,\ldots,p_n,h)$. Then in $\bar{A}[x,y]$ we have $\bar{D} = \partial/\partial\bar{G}$, using $\bar{A}[y_1,\ldots,y_n,z] = \bar{A}[\bar{p}_1,\ldots,\bar{p}_n,\bar{G}]$. Therefore we have $\ker(\bar{D}) = \bar{A}[\bar{p}_1,\ldots,\bar{p}_n]$.

We have $\bar{D}(\bar{H}) = \overline{\det(J(p,H))} = \overline{a^m} = 0$, so $\bar{H} \in \ker(\bar{D})$, so there is an $R \in A\ [y_1,\ldots,y_n]$ with $\bar{H} = \bar{R}(\bar{p})$, say $H = R(p) + a^m Q(y,z)$ for some $Q(y,z) \in A[y,z]$. It will turn out that $(p,Q) \in \mathrm{Aut}_A A[y,z]$. Namely, over $A_a$ we have $(p,H) = (p,R(p)+a^m Q)$ is invertible, so $(p,a^m Q)$ is invertible, and therefore $(p,Q)$ also and $\det J(p,Q) = a^{-m} \det J(p,H) = 1$. Because $(p,Q) \in A[y,z]$, $A \subseteq A_a$ we know by Lemma 1.1.8 of [9] that $(p,Q) \in \mathrm{Aut}_A A[y,z]$. □

In Theorem 4.2 we suppose that $a$ is not a zerodivisor. The theorem is easy to prove when $a$ is nilpotent.

**Conjecture 4.4.** *The result of* 4.2 *holds even when $a$ is a zerodivisor.*

The difficulty when $a$ is a zerodivisor is, that $A$ no longer is a subring of $A_a$, which is needed in the proof of 4.2.

## 5. Coordinates in three (or more) variables

Now one may wonder if 3.1 can be generalized to three or more variables, i.e. one considers the following question. Let $n \geqslant 2$, $a \in A$ and $\bar{A} := A/Aa$. If $p$ is a coordinate in $A[y_1,\dots,y_n,z]$ such that $\bar{p}(y_1,\dots,y_n,0)$ is a coordinate in $\bar{A}[y_1,\dots,y_n]$, does it then follow that $p(y_1,\dots,y_n,az)$ is a coordinate in $A[y_1,\dots,y_n,z]$?

In general the answer is no.

**Example 5.1.** Let $A := \mathbb{R}[A,B,C]/I$, with $I := (A^2+B^2+C^2-1)$, the coordinate ring of the 2-sphere in $\mathbb{R}^3$. Define $a := A+I$, $b := B+I$, $c := C+I$. Let $p := ay_1+by_2+z$. Then $p$ is a coordinate in $A[y_1,y_2,z]$. Writing $\bar{A} = A/(c)$, $\bar{p}(y_1,y_2,0)$ is a coordinate in $\bar{A}[y_1,y_2]$, whose mate is $-\bar{b}y_1+\bar{a}y_2$. On the other hand, it is well known that $p(y_1,y_2,cz) = ay_1+by_2+cz$ is not a coordinate in $A[y_1,y_2,z]$ (see [11, Theorem 11]). However one can prove that $p$ is a 1-stable coordinate, i.e. a coordinate in a polynomial ring with one extra variable added. More precisely, we have Proposition 5.3. To make the statement of this proposition fully clear, we need the following definition.

**Definition 5.2.** For $k \in \mathbb{N}^*$, a polynomial $f \in A[x] := A[x_1,\dots,x_n]$ is called a *$k$-stable coordinate in* $A[x_1,\dots,x_n]$ if for some new variables $y := (y_1,\dots,y_k)$, $f \in A[x,y]$ is a coordinate.

**Proposition 5.3.** *Let $A$ be any ring, $a$ a non-zerodivisor in $A$ and $\bar{A} := A/Aa$. Let $p(y_1,\dots,y_n,z)$ be a coordinate in $A[y,z] := A[y_1,\dots,y_n,z]$. If $\bar{p}(y_1,\dots,y_n,0)$ is a coordinate in $\bar{A}[y]$, then $p(y_1,\dots,y_n,az)$ is an $(n-1)$-stable coordinate in $A[y,z]$.*

**Proof.** We can choose $f_i \in A[y,z]$ and $g_i \in A[y]$ such that $(p,f_2,\dots,f_{n+1}) \in \mathrm{Aut}_A A[y,z]$ and $(\bar{p}(y,0),\bar{g}_2,\dots,\bar{g}_n) \in \mathrm{Aut}_{\bar{A}}\bar{A}[y]$. Let $\eta$ be the nilradical of $A$. We only need to prove that $p(y,az)$ is an $(n-1)$-stable coordinate over $A/\eta$. Using the obvious map $A/Aa \to (A/\eta)/(A/\eta)(a+\eta)$ we see that the assumptions are true modulo $\eta$, so we may assume that $A$ is reduced. Now we have $A[y,z]^* = A^*$. Write $F^{(1)} = (p,f) \in \mathrm{Aut}_A A[y,z]$ and $d_0 := \det JF^{(1)} \in A^*$. Then $F^{(2)} := (p,f)\circ(y,az) = (p(y,az),f(y,az)) \in \mathrm{Aut}_{A_a} A_a[y,z]$ with $\det JF^{(2)} = d_0\cdot a$.

Write $w = (w_2,\dots,w_n)$ (a $w_i$ for every $g_i$) and

$$H^{(1)} = (F^{(2)},w) = (p(y,az),f(y,az),w) \in \mathrm{Aut}_{A_a} A_a[y,z,w].$$

Then $\det JH^{(1)} = d_0 a$. Write

$$\begin{aligned}H^{(2)} &= H^{(1)}\circ(y,z,aw+g(y))\\ &= (p(y,az),f(y,az),aw+g(y)) \in \mathrm{Aut}_{A_a} A_a[y,z,w],\end{aligned}$$

then $\det JH^{(2)} = d_0 a\cdot a^{n-1} = d_0 a^n$.

Because $(\bar{p}(y,0),\bar{g}) \in \mathrm{Aut}_{\bar{A}}\bar{A}[y]$ (remember $\bar{A} = A/Aa$) we can choose $h_i \in A[y]$, for $i = 2,\dots,n+1$, such that

$$h_i(p(y,0),g(y)) \equiv f_i(y,0) \bmod (a),$$

so

$$h_i(p(y,az),aw+g(y)) \equiv f_i(y,az) \bmod (a),$$

so we can define polynomials in $A[y,z,w]$:

$$\tilde{h}_i(y,z,w) := \frac{1}{a}[f_i(y,az) - h_i(p(y,az),aw+g(y))] \in A[y,z,w].$$

Then

$$H^{(3)} := (p(y,az), \tilde{h}(y,z,w), aw + g(y)) \in \mathrm{Aut}_{A_a} A_a[y,z,w]$$

is an automorphism over $A_a$ and $\det JH^{(3)} = d_0a^n/a^n = d_0 \in A^*$. By construction $H^{(3)} \in \mathrm{End}_A A[y,z,w]$. Since $a$ is a non-zerodivisor in $A$ we have $A \subseteq A_a$. Therefore by lemma L.1.8 of [9] $H^{(3)} \in \mathrm{Aut}_A A[y,z,w]$. □

**Remark 5.4.** It is not difficult to extend 5.3 to the case that $s \leqslant n$ and $p = (p_1, \ldots, p_s)$ is a partial coordinate system. The conclusion is that $p(y,az)$ is an $(n-s)$-stable partial coordinate system.

Under some special conditions on the ring $A$ we can generalize 3.1 to the case of a coordinate of co-length 2 instead of co-length 1. The following result is a generalization of Theorem 8.7 of [15], where the case $A = \mathbb{C}[x]$ is proved. First we need a definition: if $A$ is a discrete valuation ring with maximal ideal $(t)$, then a subfield $k$ of $A$ is called a *field of representatives* of $A$ if the natural homomorphism $A \to A/(t)$ restricts to an isomorphism $k \xrightarrow{\sim} A/(t)$. The main result we prove is

**Theorem 5.5.** *Let $A$ be a $\mathbb{Q}$-algebra and a Dedekind domain such that for each maximal ideal $\mathfrak{m}$ of $A$ the discrete valuation ring $A_{\mathfrak{m}}$ has a field of representatives. Let $a \in A$, $p(y_1, y_2, z)$ a coordinate in $A[y_1, y_2, z]$ and $\bar{A} := A/Aa$. If $\bar{p}(y_1, y_2, 0)$ is a coordinate in $\bar{A}[y_1, y_2]$, then $p(y_1, y_2, az)$ is a coordinate in $A[y_1, y_2, z]$.*

**Proof.** Since $A$ is a Dedekind domain it is noetherian of dimension one. So by Theorem V.3.2 in [1] $A$ is Hermite, which by Theorem 2.6 implies that in order to show that $p(y_1, y_2, az)$ is a coordinate in $A[y_1, y_2, z]$ it suffices to prove that $p(y_1, y_2, az)$ is a coordinate in $A_{\mathfrak{m}}$, for each maximal ideal $\mathfrak{m}$ of $A$. Since all assumptions of our theorem are true when considered over $A_{\mathfrak{m}}$, we may assume that $A$ is a discrete valuation ring with maximal ideal $(t)$. By our hypothesis we may assume additionally that $A$ contains a field of representatives. Then the statement follows from the next Proposition 5.6 and Theorem 5.7. □

Another definition: if $R$ is a ring and $p(y_1, y_2, z) \in R[y_1, y_2, z]$, we call $p$ a *z-coordinate* if it is a coordinate in the polynomial ring $R[z][y_1, y_2]$ in two variables over $R[z]$.

**Proposition 5.6.** *Let $A$ be any ring, $t$ a non-zerodivisor of $A$ and suppose there is a subring $R$ of $A$ such that the restriction $\phi : R \hookrightarrow A \to A/(t)$ is an isomorphism. Let $p(y_1, y_2, z)$ be a coordinate in $A[y_1, y_2, z]$ and write $\bar{A} = A/(t)$. If $\bar{p}(y_1, y_2, z)$ is a z-coordinate in $\bar{A}[z][y_1, y_2]$, then $p(y_1, y_2, t^k z)$ is a coordinate in $A[y_1, y_2, z]$ for all $k \in \mathbb{N}$.*

**Proof.** We are going to use induction on $k$. Suppose we know the proposition for $k = 1$, then $p(y_1, y_2, tz)$ is a coordinate. Modulo $(t)$, we have $\bar{p}(y_1, y_2, tz) = \bar{p}(y_1, y_2, 0)$. This is a $z$-coordinate, which can be seen using the fact that $\bar{p}(y_1, y_2, z)$ is a $z$-coordinate.

Namely, call its mate $\bar{q}(y_1, y_2, z)$, then there is an $r \in A[z][y_1, y_2]$ with

$$y_1 = \bar{r}(\bar{p}(y_1, y_2, z), \bar{q}(y_1, y_2, z)) = \bar{r}(\bar{p}(y_1, y_2, 0), \bar{q}(y_1, y_2, 0)).$$

The last equation follows by substituting $z := 0$ on both sides. In the same way, we can make $y_2$. So $\bar{p}(y_1, y_2, 0)$ is a $z$-coordinate over $A/(t)$.

Therefore, we may add another $t$ in $p(y_1, y_2, tz)$ (applying the proposition for $k=1$), so $p(y_1, y_2, t^2z)$ is a coordinate. Repeating this we conclude that $p(y_1, y_2, t^kz)$ is a coordinate for each $k \in \mathbb{N}$. The remaining case therefore is $k = 1$.

Let $\alpha \in \mathrm{Aut}_A A[y_1, y_2, z]$ with $\alpha(y_1) = p(y_1, y_2, z)$. Let $\phi$ be the isomorphism $R \hookrightarrow A \to \bar{A}$. We extend $\phi$ to the isomorphism $\phi : R[y_1, y_2, z] \xrightarrow{\sim} \bar{A}[y_1, y_2, z]$.

The Greek letters will be treated as ring homomorphisms in the compositions. Let

$$\alpha_0 := \phi^{-1}\bar{\alpha}\phi \in \mathrm{Aut}_R R[y_1, y_2, z] \subset \mathrm{Aut}_A A[y_1, y_2, z].$$

We also have a $z$-automorphism $\beta$ of $\bar{A}[y_1, y_2, z]$ of which $\bar{p}(y_1, y_2, z)$ is the first component. Let

$$\beta_0 := \phi^{-1}\beta\phi \in \mathrm{Aut}_R R[y_1, y_2, z] \subset \mathrm{Aut}_A A[y_1, y_2, z].$$

Using $\phi(y_1) = y_1$, we have

$$\alpha_0(y_1) = \phi^{-1}(\bar{\alpha}(\phi(y_1))) = \phi^{-1}(\bar{p}(y_1, y_2, z)) = \phi^{-1}(\beta(\phi(y_1)) = \beta_0(y_1),$$

so $\alpha_0^{-1}(\beta_0(y_1)) = y_1$. Now define

$$\gamma = \alpha \circ \alpha_0^{-1} \circ \beta_0 \in \mathrm{Aut}_A A[y_1, y_2, z].$$

Then $\gamma(y_1) = \alpha(y_1) = p(y_1, y_2, z)$ and $\overline{\gamma(z)} = \overline{\alpha(\alpha_0^{-1}(z))} = z$, since $\bar{\alpha} = \bar{\alpha}_0$. Say $\gamma(z) = z + t \cdot h(y_1, y_2, z)$.

Define $\sigma = (y_1, y_2, tz) \in \mathrm{Aut}_{A_t} A_t[y_1, y_2, z]$. Because $t$ is not a zerodivisor, we have $A \subseteq A_t$. Then $\sigma\gamma\sigma^{-1}(y_1) = p(y_1, y_2, tz)$, $\sigma\gamma\sigma^{-1}(y_2) = \sigma\gamma(y_2) \in A[y_1, y_2, z]$ and

$$\sigma\gamma\sigma^{-1}(z) = \sigma\gamma\left(\frac{z}{t}\right) = \frac{1}{t}\,\sigma\gamma(z) = \frac{1}{t}\,\sigma(z + t \cdot h(y_1, y_2, z))$$

$$= \frac{1}{t}(tz + th(y_1, y_2, tz) = z + h(y_1, y_2, tz) \in A[y_1, y_2, z],$$

so actually $\sigma\gamma\sigma^{-1} \in \mathrm{End}_A A[y_1, y_2, z]$. Furthermore, $\det J\gamma(0) \in A^*$, $\det J\sigma = t$ and $\det J\sigma^{-1} = 1/t$, so $\det J(\sigma\gamma\sigma^{-1})(0) = 1/t \cdot \det J\gamma(\sigma^{-1}(0)) \cdot t = \det J\gamma(0) \in A^*$. Now we apply Lemma 1.1.8 of [9] to see that $\sigma\gamma\sigma^{-1}$ is an invertible map over $A$ with first component $p(y_1, y_2, tz)$. $\square$

To conclude this paper and the proof of 5.5, we show

**Theorem 5.7.** *Let $R$ be a $\mathbb{Q}$-algebra and $p(y_1, y_2, z) \in R[y_1, y_2, z]$. Suppose $R[y_1, y_2, z]/(p(y_1, y_2, z)) \cong R^{[2]}$, then the following statements are equivalent*:

(i) *$p(y_1, y_2, 0)$ is a coordinate in $R[y_1, y_2]$*,

(ii) *$R[y_1, y_2]/(p(y_1, y_2, 0) \cong R^{[1]}$*,

(iii) *$p(y_1, y_2, z)$ is a $z$-coordinate.*

**Proof.** Implication (i) $\Rightarrow$ (ii) is easy to see. For (iii) $\Rightarrow$ (i) see the beginning of the proof of 5.6. It remains to prove (ii) $\Rightarrow$ (iii). By the hypothesis on $p$ there exists an isomorphism

$$\phi : R[z, y_1, y_2]/(p(y_1, y_2, z)) \xrightarrow{\sim} R^{[2]}$$

and therefore we have

$$R[z, y_1, y_2]/(p(y_1, y_2, z), z) \cong R^{[2]}/\phi(z).$$

Using the given isomorphism in assumption (ii) in the last '$\cong$', we get

$$R^{[2]}/\phi(z) \cong R[z, y_1, y_2]/(p(y_1, y_2, z), z) \cong R[y_1, y_2]/(p(y_1, y_2, 0)) \cong R^{[1]}.$$

Now we use the extra assumption $\mathbb{Q} \subset A$ in applying 2.5 and we see that $\phi(z)$ is a coordinate in $R^{[2]}$, so there is an automorphism of $R^{[2]} = R[z, T]$ that sends $\phi(z)$ to $z$. Combining this with $\phi$ we get an isomorphism

$$\psi : R[z, y_1, y_2]/(p(y_1, y_2, z)) \xrightarrow{\sim} R[z, T], \quad \psi(z) = z$$

or, writing $A = R[z]$, we have

$$A[y_1, y_2]/(p(y_1, y_2, z)) \cong A[T],$$

so using 2.5 again, we see that $p(y_1, y_2, z)$ is a $z$-coordinate over $R$. $\square$

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